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(6) A COMPLETE IMPORTANCE RANKING FOR COMPONENTS OF BINARY COHERENT SYSTEMS, WITH EXTENSIONS TO MULTI-STATE SYSTEMS,

by

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A COMPLETE IMPORTANCE RANKING FOR COMPONENTS OF BINARY COHERENT SYSTEMS, WITH EXTENSIONS TO MULTI-STATE SYSTEMS

D. A. Butler

1. Introduction

Given a system composed of many components, a question of considerable interest is which components are most relevant or crucial to the proper functioning of the system. In response to this question, a number of importance measures and rankings have been proposed (3), (4), (5), (9). This paper investigates a new ranking and compares it to existing rankings, principally the ranking induced by the Birnbaum reliability importance measure. The new ranking is based upon minimal cuts and provides a complete ordering of all the system's components relative to their importance to the system reliability. This ranking has three main points in its favor, (i) the calculations involved require only readily obtainable information; (ii) the calculations are usually quite simple; and (iii) the ranking is designed for use with systems consisting of highly reliable components, the most common case.

The final section of the paper deals with extensions of importance measures and rankings to systems in which both the system and its components may be in any of a finite number of states. Many of the results about importance measures and rankings for binary systems

established in preceding sections are shown to extend to the more sophisticated multi-state systems. Also, the multi-state importance measures and rankings are shown to be decomposable into a number of sub-importance measures and rankings.

2. Preliminary Definitions

Consider a system which consists of n components labelled $1, \ldots, n$.* With each component i, associate the binary random variable X_i, where

$$X_i = \begin{cases} 1 & \text{if component } i \text{ functions satisfactorily,} \\ 0 & \text{otherwise.} \end{cases}$$

The time domain over which the components and the system are to function is implicit and not specified. This time may be fixed or variable and may not even be the same for each component.

Let $\underline{X} = (X_1, \ldots, X_n)$. It will be assumed that the system functions or fails depending solely upon the random outcome $\underline{X}$. Therefore one can define a function ϕ as follows:

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if the system functions given } \underline{X} = \underline{x} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } \underline{x} \in S,$$

where $S = \{\underline{x} \in \mathbb{R}^n : x_i = 0, 1\}$ is the range of $\underline{X}$.

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*The notation and basic definitions of Sections 2, 3 and 4 follow that of (2) throughout.

This function ϕ is called the structure function and relates the system state (namely 0 or 1) to the component states.

For each $\underline{x} \in \mathbb{R}^n$, let $(1_i,\underline{x})$ denote the vector $(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$. Similarly, let $(0_i,\underline{x})$ denote the vector $(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$.

Definition 1. A component i is relevant if and only if there exists an $\underline{x} \in S$ such that $\phi(0_i,\underline{x}) \neq \phi(1_i,\underline{x})$.* Otherwise, component i is irrelevant.

A component is irrelevant simply if it never affects the state of the system. Of course, most systems do not contain irrelevant components. Also, most systems have the properties that (i) the system fails if all its components fail and functions if all its components functio , and (ii) it is not possible to degrade the system state by upgrading one or more component states. These fundamental properties are embodied in the following definition.

Definition 2. A structure function ϕ is coherent if and only if

i) $\phi(\underline{0}) = 0$, $\phi(\underline{1}) = 1$,

ii) $\phi(\underline{x})$ is non-decreasing in $\underline{x}$,

iii) each component is relevant.

The ordered pair (N,ϕ) where N is the set of component indices is called a coherent system.

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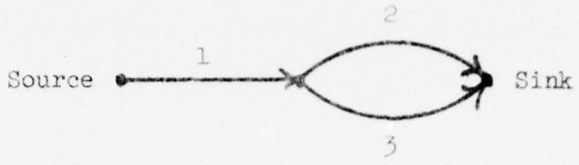
*To avoid the use of multiple parentheses, the notation $\phi(0_i,\underline{x})$ will be used in place of $\phi((0_i,\underline{x}))$. When no confusion can result similar simplifications will be employed without mention in the following.

Example 1:

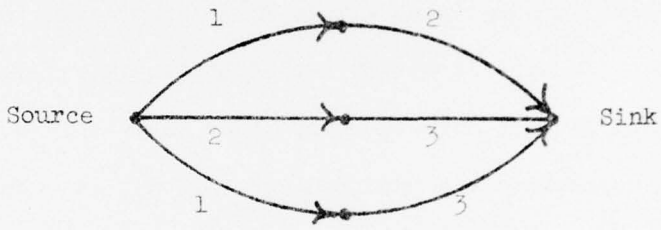
a) $N = \{1,2,3\}$

$$\phi(\underline{x}) = x_1(1 - (1-x_2)(1-x_3))$$



b) $N = \{1,2,3\}$

$$\phi(\underline{x}) = x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3$$



Example 1 above exhibits two simple coherent systems. The structure functions for both systems are specified by giving formulas for them. However, the accompanying graphs provide more easily understood representations of the system structures. Any coherent system is representable by such a graph. These graphs consist of directed and undirected edges and always contain a unique source and a unique sink. A specific component is associated with each edge. The system functions if and only if it is possible to proceed from source to sink along a path (following the orientation of any directed edges in the path) all of

whose edges correspond to functioning components. In Example 1(a) it is seen that if components 1 and 2 function and component 3 fails, then the system functions because it is possible to go from the source to the sink along the "functioning" path (1,2). However, should component 1 fail, then regardless of the states of components 2 and 3, the system fails because no such path then exists. The coherent system of Example 1(b) is called a 2-out-3 system. This system consists of three components and functions if and only if at least two of the components function. Note that in the graphical representation of this system each component is assigned to more than one edge. It is not possible to represent this system by a graph for which there is a one-to-one correspondence between edges and components.

In the following, it will be assumed that the random variables X_i are mutually independent. (Of course for many systems such is not the case. Factors which cause dependence include the "loading" which may occur on remaining components should one or more parts fail, and "common-mode" failures, that is, simultaneous failures of two or more components stemming from a single cause.) Let $p_i = \Pr\{X_i=1\}$; p_i is called the reliability of component i. Similarly, the system reliability is defined as $\Pr\{\phi(\underline{X}) = 1\}$. Assuming independence, one can compute the system reliability knowing only the value of $\underline{p} = (p_1, p_2, \ldots, p_n)$,* and thus define the reliability function, $h(\underline{p}) = \Pr\{\phi(\underline{X}) = 1\}$.

*In the following whenever the scalar p appears in an expression normally involving the vector $\underline{p}$, $\underline{p}$ will be understood to be a vector all of whose components are equal to p, i.e., $\underline{p} = (p,p,\ldots,p)$.

Example 1 (continued):

(a) $h(\underline{p}) = p_1(1 - (1-p_2)(1-p_3))$

(b) $h(\underline{p}) = p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3$

3. Definitions of Component Importance--Binary Systems

Consider an arbitrary coherent system (N,ϕ) of independent components with associated reliability function $h(\underline{p})$. Birnbaum (4) defines importance as follows.

Definition 3 (Birnbaum): The reliability importance of component i, $I_h(i;\underline{p})$, is $\partial h(\underline{p})/\partial p_i$. (Often the dependence upon $\underline{p}$ is suppressed and the notation simplified to $I_h(i)$.)

An equivalent definition of $I_h(i)$ is given by Proposition 1 below, due to Birnbaum (4).

Proposition 1: $$I_h(i;\underline{p}) = h(1_i,\underline{p}) - h(0_i,\underline{p})$$

Proof: By conditioning on whether or not $X_i = 1$, one can easily show that

$$h(\underline{p}) = p_i h(1_i,\underline{p}) + (1-p_i)\, h(0_i,\underline{p}) \quad \text{for all } \underline{p}. \tag{1}$$

The result then follows immediately upon differentiating.

□

Proposition 1 simply states that the Birnbaum reliability importance of component i is the probability that the system functions given component i functions minus the probability that the system functions given component i fails. Clearly, $0 \leq I_h(i;\underline{p}) \leq 1$, and $I_h(i;\underline{p})$ does not depend upon p_i. Because of this last property, $I_h(i)$ can be used to directly calculate the change in the system reliability resulting from a change in p_i.

Proposition 2: Choose $\delta \in \mathbb{R}$ so that $0 \leq p_i + \delta \leq 1$. Define $\hat{\underline{p}} \in \mathbb{R}^n$ by

$$\hat{\underline{p}}_j = \begin{cases} p_j & j \neq i \\ p_i + \delta & j = i. \end{cases}$$

Then $h(\hat{\underline{p}}) = h(\underline{p}) + \delta I_h(i;\underline{p})$.

Proof. By Proposition 1 and equation (1),

$$\begin{aligned} h(\underline{p}) + \delta I_h(i;\underline{p}) &= p_i h(1_i,\underline{p}) + (1-p_i)\, h(0_i,\underline{p}) + \delta h(1_i,\underline{p}) - \delta h(0_i,\underline{p}) \\ &= \hat{p}_i h(1_i,\underline{p}) + (1-\hat{p}_i)\, h(0_i,\underline{p}) \\ &= \hat{p}_i h(1_i,\hat{\underline{p}}) + (1-\hat{p}_i)\, h(0_i,\hat{\underline{p}}) \\ &= h(\hat{\underline{p}}). \end{aligned}$$

□

To calculate the Birnbaum reliability importance of a component, a knowledge of the reliability function $h(\cdot)$ and the component reliabilities $p_1, p_2, \ldots, p_n$ are required. Birnbaum has also proposed the following definition of structural importance, which can be used when the component reliabilities are unknown.

Definition 4: The vector $\underline{x} \in S$ is a critical vector for component i if and only if $\phi(1_i,\underline{x}) - \phi(0_i,\underline{x}) = 1$.

Definition 5: Let $n_\phi(i) = |\{\underline{x} \in S:\underline{x}$ is a critical vector for component i$\}|$, where $|\cdot|$ denotes set cardinality. The Birnbaum structural importance of component i, $I_\phi(i)$, is defined by $I_\phi(i) = 2^{-n}\, n_\phi(i)$, or equivalently, by

$$I_\phi(i) = 2^{-n} \sum_{\underline{x}\in S} [\phi(1_i,\underline{x}) - \phi(0_i,\underline{x})] .$$

This definition of component importance requires only a knowledge of the structure function for its calculation. Birnbaum's reliability and structural importance measures are related in the following proposition.

Proposition 3 (Birnbaum): $I_\phi(j) = I_h(j;\, 1/2)$.

Proof: By summing over the 2^n possible values for the random vector $\underline{X}$,

$$h(\underline{p}) = \sum_{\underline{x}\in S} p_i^{x_i}(1-p_i)^{1-x_i}\, \phi(\underline{x}) .$$

Thus

$$I_h(j;\underline{p}) = \partial h(\underline{p})/\partial p_j = \frac{1}{2} \sum_{\underline{x}\in S} \Big[\prod_{i\neq j} p_i^{x_i}(1-p_i)^{1-x_i} \Big][\phi(1_j,\underline{x}) - \phi(0_j,\underline{x})].$$

$$I_h(j;1/2) = \sum_{\underline{x}\in S} 2^{-n}[\phi(1_j,\underline{x}) - \phi(0_j,\underline{x})] .$$

□

Thus, the Birnbaum structural importance is just the Birnbaum reliability importance where each component is assumed to be as likely to fail as to function. A more complete discussion of the Birnbaum reliability and structural importance measures is contained in (4).

Barlow and Proschan (3) have also developed structural and reliability importance measures. Their reliability importance measure requires a knowledge of the distribution F_i of the time-to-failure for each component i.

Definition 6 (Barlow-Proschan): The B-P reliability importance of component i, $P_h(i)$ is given by

$$P_h(i) = \int_0^\infty [h(1_i, \underline{\bar{F}}(t)) - h(0_i, \underline{\bar{F}}(t))] \, dF_i(t) ,$$

or equivalently,

$$P_h(i) = \int_0^\infty I_h(i; \underline{\bar{F}}(t)) \, dF_i(t) .$$

[Note: $\underline{\bar{F}}(t)$ denotes the vector $(1-F_1(t), 1-F_2(t), \ldots, 1-F_n(t))$ in the above.] This measure can be interpreted as the probability that the failure of component i causes the system to fail (3). When the F_i are unknown, Barlow and Proschan suggest the use of the above measure with the substitution of a common time-to-failure distribution F for each F_i. By making the change of variable $p = \bar{F}(t)$ the following definition results.

Definition 7 (Barlow-Proschan): The B-P structural importance of component i, $P_\phi(i)$, is given by

$$P_\phi(i) = \int_0^1 [h(1_i,p) - h(0_i,p)]dp$$

or equivalently

$$P_\phi(i) = \int_0^1 I_h(i;p)dp \ .$$

(Note scalar p in above.)

The two importance measures $I_\phi(i)$ and $P_\phi(i)$ are termed structural measures because they only require a knowledge of the system structure function to be calculated. This gives them an important practical advantage over the more sophisticated reliability importance measures, because often the more detailed knowledge required for the calculation of these latter measures is unobtainable. Both structural measures can be derived from the Birnbaum reliability importance measure, $I_h(i;p)$. The Birnbaum structural importance measure assumes a common reliability of one-half for each component, while the B-P structural measure averages $I_h(i;p)$ over all $p \in [0,1]$. Thus both measures favor neither high nor low component reliabilities. This may represent a significant weakness of these measures because often component reliabilities, while unknown, are thought to be high (perhaps even .99 or higher). It would seem desirable, therefore, to develop a measure or ranking that is structural (i.e., is based solely upon the system

structure function and therefore not upon $\underline{p}$), yet is somehow related to the Birnbaum reliability importance measure for high values of $\underline{p}$. The new ranking proposed in this paper is such a result. This new ranking is based upon cuts.

Definition 8: A set $C \subset N$ is a cut if and only if $\phi(\underline{x}) = 0$ for every $\underline{x} \in S$ such that $x_i = 0$ for each $i \in C$. A cut set C is minimal if and only if no proper subset of C is a cut.

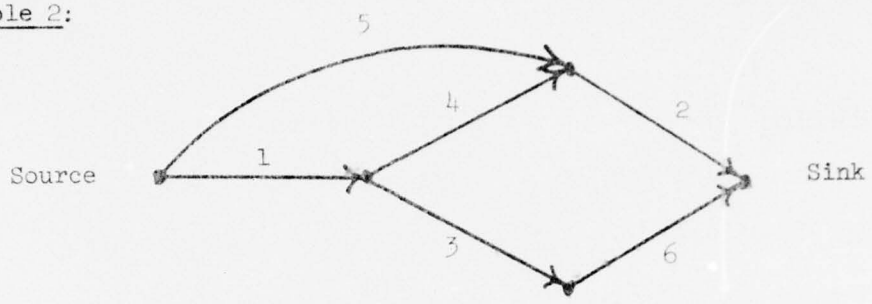
The notions of cuts and minimal cuts appear widely in reliability theory. A knowledge of all the minimal cuts of a coherent system (N,ϕ) is equivalent to a knowledge of ϕ since $\phi(\underline{x}) = 1$ if and only if $\{i:x_i = 0\}$ contains no minimal cuts. A methodology for computing all minimal cuts for an arbitrary coherent system expressed as a fault tree appears in (6).

For each component ℓ of a coherent system (N,ϕ) with t minimal cuts, let $d_{ij}^{(\ell)}$ denote the number of unions of i distinct min cuts that contain exactly j components and include component ℓ, $(1 \leq i \leq t,\ 1 \leq j \leq n)$. Let $b_j^{(\ell)} = \sum_{i=1}^{t} (-1)^{i-1} d_{ij}^{(\ell)}$. Let $\underline{b}^{(\ell)} = (b_1^{(\ell)}, b_2^{(\ell)}, \ldots, b_n^{(\ell)})$.

Definition 9: Component ℓ is more cut-important than component k, denoted $\ell >_c k$, if and only if $\underline{b}^{(\ell)} \rangle \underline{b}^{(k)}$, where $\rangle$ denotes lexicographic ordering. Components ℓ and k are equally cut-important, denoted $\ell =_c k$, if and only if $\underline{b}^{(\ell)} = \underline{b}^{(k)}$.

This ranking, which provides a complete ordering of all components, is an extension of a partial ranking developed in an earlier paper (5).

Example 2:



$$\phi(\underline{x}) = 1 - [1 - x_1x_2x_4][1 - x_1x_3x_6][1 - x_2x_5]$$

Min Cuts: $C_1 = \{1,5\}$, $C_2 = \{2,3\}$, $C_3 = \{2,6\}$, $C_4 = \{4,5,6\}$, $C_5 = \{3,4,5\}$, $C_6 = \{1,2\}$.

For $\ell = 2$, the non-zero $d_{ij}^{(\ell)}$'s are as follows: $d_{12}^{(2)} = 3$, $d_{23}^{(2)} = 4$, $d_{24}^{(2)} = 4$, $d_{25}^{(2)} = 4$, $d_{34}^{(2)} = 3$, $d_{35}^{(2)} = 11$, $d_{36}^{(2)} = 5$, $d_{45}^{(2)} = 4$, $d_{46}^{(2)} = 11$, $d_{56}^{(2)} = 6$, $d_{66}^{(2)} = 1$. Thus $\underline{b}^{(2)} = (0,3,-4,-1,3,-1)$. Similarly,

$$\underline{b}^{(1)} = (0,2,-3,-1,3,-1), \quad \underline{b}^{(3)} = (0,1,-1,-2,3,-1),$$
$$\underline{b}^{(4)} = (0,0,2,-5,4,-1), \quad \underline{b}^{(5)} = (0,1,1,-5,4,-1), \qquad (2)$$
$$\underline{b}^{(6)} = (0,1,-1,-2,3,-1).$$

Therefore $2 >_c 1 >_c 5 >_c 3 =_c 6 >_c 4$.

$$h(\underline{p}) = p_1p_2p_4 + p_2p_5 + p_1p_3p_6 - p_1p_2p_4p_5 - p_1p_2p_3p_4p_6$$
$$- p_1p_2p_3p_5p_6 + p_1p_2p_3p_4p_5p_6 .$$

$$I_h(1;p) = 2p^2 - p^3 - 2p^4 + p^5 , \quad I_h(2;p) = p + p^2 - p^3 - 2p^4 + p^5,$$
$$I_h(3;p) = p^2 - 2p^4 + p^5 , \quad I_h(4;p) = p^2 - p^3 - p^4 + p^5 ,$$
$$I_h(5;p) = p - p^3 - p^4 + p^5 , \quad I_h(6;p) = p^2 - 2p^4 + p^5 .$$

$$I_\emptyset(1) = 9/32, \quad I_\emptyset(2) = 17/32, \quad I_\emptyset(3) = 5/32,$$
$$I_\emptyset(4) = 3/32, \quad I_\emptyset(5) = 11/32, \quad I_\emptyset(6) = 5/32 .$$

$$P_\emptyset(1) = .183, \quad P_\emptyset(2) = .350, \quad P_\emptyset(3) = .100,$$
$$P_\emptyset(4) = .050, \quad P_\emptyset(5) = .217, \quad I_\emptyset(6) = .100,$$

Denote the ordering of the components induced by $P_\emptyset(i)$ by $>_p$. Then

$$2 >_p 5 >_p 1 >_p 3 =_p 6 >_p 4 .$$

Denote the ordering of the components induced by $I_\emptyset(i)$ by $>_\emptyset$. Then

$$2 >_\emptyset 5 >_\emptyset 1 >_\emptyset 3 =_\emptyset 6 >_\emptyset 4.$$

Finally, denote the ordering of the components induced by $I_h(\cdot;p)$ by $>_h$. (This ordering depends upon the value of p.) For any value of p,

$$2 >_h 5,\ 1 >_h 3 =_h 6 >_h 4 .$$

For $p > (-1 + 5)/2 \approx .618$, $1 >_h 5$. For $p < (-1 + 5)/2$, $5 >_h 1$. Thus when $p < .618$ the ordering induced by $I_h(\cdot;p)$ gives the same component ordering for this example as do the other two measures, $I_\phi(\cdot)$ and $P_\phi(\cdot)$. For $p > .618$, however, the ordering $>_h$ reverses 1 and 5 relative to these other orderings. For these higher values of p, the orderings $>_h$ and $>_c$ are identical. In the next section this last statement will be proved as a general result about coherent systems of independent components.

4. Analysis of the Cut-Importance Ranking--Binary Systems

As stated in the introduction, the cut importance ranking has three main favorable properties; i) it is based upon readily obtainable information, ii) is usually easily calculated, and iii) is in some sense biased for high component reliabilities. The first property is already established, since this ordering is based only upon the system structure function through the minimal cuts of the system. This section will discuss the second and third properties.

The precise meaning of the third property of the cut-importance ranking is given in Theorems 1 and 2 below. The first theorem relates the cut-importance ranking to the Birnbaum reliability importance measure in the case where the component reliabilities are equal and high.

Theorem 1: For p sufficiently close to one, the orderings $>_h$ and $>_c$ are identical.

Proof: The above is a direct result of Lemma 1 which follows. Using the lemma, it is clear that $\ell =_c k$ if and only if $I_h(\ell;p) = I_h(k;p)$ for all p. Also, $\ell >_c k$ if and only if $\underline{b}^{(\ell)} - \underline{b}^{(k)} \succ \underline{0}$, and $\underline{b}^{(\ell)} - \underline{b}^{(k)} \succ \underline{0}$ if and only if $I_h(\ell;p) > I_h(k;p)$ for all p sufficiently close to one. □

Lemma 1:
$$I_h(\ell;p) = \sum_{j=1}^{n} b_j^{(\ell)} (1-p)^{j-1} .$$

Proof:
$$h(\underline{p}) = \Pr\left(\bigcap_{i=1}^{t} E_i \right) ,$$

where E_i denotes the event that at least one component in i^{th} min cut functions. Thus

$$h(\underline{p}) = 1 - \Pr\left(\bigcup_{i=1}^{t} E_i^c \right).$$

By the inclusion-exclusion principle (7),

$$h(\underline{p}) = 1 - \sum_{i=1}^{t} (-1)^{i-1} S_i,$$

where

$$S_i = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} \Pr(E_{j_1}^c \cap E_{j_2}^c \cap \cdots \cap E_{j_i}^c) .$$

Now using the independence assumption,

$$S_i = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} \left[\prod_{k \in C_{j_1} \cup C_{j_2} \cup \cdots \cup C_{j_i}} (1-p_k) \right]$$

where $C_1, \ldots, C_t$ are the minimal cuts of the system. Thus

$$I_h(\ell;\underline{p}) = \frac{\partial h(\underline{p})}{\partial p_\ell} = \sum_{i=1}^{t} (-1)^{i-1} \sum_{\substack{1 \le j_1 < j_2 < \cdots < j_i \le t \\ \ell \in C_{j_1} \cup C_{j_2} \cup \cdots \cup C_{j_i}}} \left[\prod_{\substack{k \in C_{j_1} \cup C_{j_2} \cup \cdots \cup C_{j_i} \\ k \ne \ell}} (1-p_k) \right].$$

Recalling $p_1 = p_2 = \cdots = p_n = p$, and the definition of $d_{ij}^{(\ell)}$,

$$I_h(\ell;p) = \sum_{i=1}^{t} \sum_{j=1}^{n} (-1)^{i-1} (1-p)^{j-1} d_{ij}^{(\ell)} ,$$

$$= \sum_{j=1}^{n} b_j^{(\ell)} (1-p)^{j-1} .$$

□

We now consider the case where the component reliabilities are high but unequal. Let $\underline{p}(\epsilon)$ be a vector-valued function of the positive scalar ϵ for which $0 < p_i(\epsilon) < 1$ for all $\epsilon \in (0,\infty)$ and $1 \le i \le n$. Let $\lim_{\epsilon \to 0} \underline{p}(\epsilon) = \underline{1}$. Unfortunately, it is not in general true that the component ordering induced by $I_h(\cdot;\underline{p}(\epsilon))$ coincides with $>_c$ for all ϵ sufficiently close to zero. However with some additional assumptions on $\underline{p}(\epsilon)$ some partial results along these lines are possible. First we establish a simple formula for the first non-zero coordinate in any vector $\underline{b}^{(\ell)}$ (Corollary 1 below).

Proposition 4: For each component k, let e_k be the cardinality of the smallest minimal cut containing component k, and let f_k be the number of minimal cuts of cardinality e_k containing k. Then i) $e_k = \min\{j: b_j^{(k)} \neq 0\}$, and ii) $f_k = b_{e_k}^{(k)}$.

Proof. By definition $d_{1e_k}^{(k)} = f_k$. Also any union of two or more minimal cuts at least one of which contains k must have cardinality at least e_k+1. Thus $d_{ie_k}^{(k)} = 0$ for all $i \geq 2$. Therefore

$$b_{e_k}^{(k)} = \sum_{i=1}^{t} (-1)^{i-1} d_{ie_k}^{(k)} = f_k .$$

Also, since component k is contained in no cuts of cardinality smaller than e_k, $d_{ij}^{(k)} = 0$ for all $j < e_k$. Thus $b_j^{(k)} = 0$ for $j < e_k$.

□

Corollary 1: (i) If $e_\ell < e_k$, then $\ell >_c k$.

(ii) If $e_\ell = e_k$ and $f_\ell > f_k$, then $\ell >_c k$.

Theorem 2: Assume that for some $M_1, M_2 \in \mathbb{R}$,

$$\frac{1}{M_1} \leq \frac{q_1(\epsilon)}{q_j(\epsilon)} \leq \frac{1}{M_2} \qquad \text{for all sufficiently small } \epsilon.$$

If either i) $e_\ell < e_k$, or ii) $e_\ell = e_k$ and $(f_\ell/f_k) > (M_1/M_2)^{e_k-1}$, then there exists an $\hat{\epsilon} > 0$ such that $I_h(\ell;\underline{p}(\epsilon)) > I_h(k;\underline{p}(\epsilon))$ for all $\epsilon < \hat{\epsilon}$.

Proof: See Theorem 2 in (5).

Further results along these lines are surely possible, but their value is questionable because the hypotheses become too complex. From a practical standpoint users of the cut-importance ranking should be aware that while the ranking can be useful even when component reliabilities are unequal, it may be misleading if the differences in the orders of magnitude of the unreliabilities are too great.

We now turn to the question of the computational complexities involved in determining the cut-importance ranking of a system's components. It is clear that the task of computing the entire vector $\underline{b}^{(k)}$ for each component k can be a formidable one for a complex system with many minimal cuts. However, Proposition 4 and Corollary 1 show that often components can be compared by only determining the easily computed quantities e_k and f_k. For instance, in Example 2 on pages 12-14, it is possible to determine that

$$2 >_c 1 >_c 5, 3, 6 >_c 4$$

in this manner. Also, since the structure function is symmetric in x_3 and x_6, it is clear that $3 =_c 6$. Thus additional calculations are necessary only to compare components 3 and 5. As can be seen from Equations (2) of Example 2, the ordering of these two components can be determined by computing the next entries in $\underline{b}^{(3)}$ and $\underline{b}^{(5)}$, namely $b_3^{(3)}$ and $b_3^{(5)}$. The last three entries in each vector $\underline{b}^{(\cdot)}$ are irrelevant for the purposes of ranking the components in this example.

In general, most components can be compared by determining the first non-zero entry in $\underline{b}^{(\cdot)}$ via Corollary 1. Other entries in $\underline{b}^{(\cdot)}$ are computed only as necessary.

Computations can also be simplified when the system under consideration contains subsystems, or "modules".

Definition 10: The coherent system (A,χ) is a module of the coherent system (N,ϕ) if and only if

$$\phi(\underline{x}) = \psi(\chi(\underline{x}^A), \underline{x}^{A^c}) \qquad \text{for all } \underline{x} \in S ,$$

where ψ is a coherent structure function and $A \subset N$. In the above, $\underline{x}^A$ denotes a vector with components x_i, $i \in A$, and $A^c = N\text{-}A$.

Intuitively, a module is a subset of components organized into some substructure and which affects the system performance only through the performance of the substructure. The following proposition, due to Birnbaum (4), gives a formula relating the Birnbaum reliability importance measures of a system and its modules.

Proposition 5: Let (A,χ) be a module of (N,ϕ) and let $I_h^{\chi}(\cdot)$, $I_h^{\phi}(\cdot)$, and $I_h^{\psi}(\cdot)$ denote the Birnbaum reliability importance measures for the structure functions χ, ϕ, and ψ, respectively. Then

$$I_h^{\phi}(k) = I_h^{\psi}(1) \cdot I_h^{\chi}(k) \qquad \text{for all } k \in A.$$

Proof: Let $h_\chi(\cdot)$, $h_\phi(\cdot)$, and $h_\psi(\cdot)$ denote the reliability functions of the structures χ, ϕ, and ψ, respectively. Then

$$h_\phi(\underline{p}) = h_\psi(h_\chi(\underline{p}^A), \underline{p}^{A^c}) .$$

Using the chain rule for differentiation yields the result. □

Corollary 2: Let $\underline{b}^{\chi\cdot}$, $\underline{b}^{\phi\cdot}$ and $\underline{b}^{\psi\cdot}$ denote the vectors $\underline{b}^{(\cdot)}$ corresponding to χ, ϕ, and ψ, respectively. Then

$$b_j^{\phi k} = \sum_{i=1}^{j} b_i^{\psi l} \, b_{j-i+1}^{\chi k} \qquad \text{for all } k \in A ,$$

where the definition of $\underline{b}^{\chi\cdot}$ is extended to include zero coordinates for $i > |A|$, and $\underline{b}^{\psi l}$ is extended similarly. (The above equation is just an expression of the fact that $\underline{b}^{\phi k}$ is the convolution of the finite sequences $\underline{b}^{\psi l}$ and $\underline{b}^{\chi k}$.)

Proof: By Lemma 1 and Proposition 5,

$$\sum_{j=1}^{n} b_j^{\phi k}(1-p)^{j-1} = \left(\sum_{j=1}^{|A^c|+1} b_j^{\psi l}(1-p)^{j-1}\right)\left(\sum_{j=1}^{|A|} b_j^{\chi k}(1-p)^{j-1}\right)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{j} b_i^{\psi l} \cdot b_{j-i+1}^{\chi k}\right)(1-p)^{j-1} .$$

Since this equality holds for all $0 \le p \le 1$, each pair of coefficients of the two polynomials must be identical. □

Corollary 2 can be applied to make the calculation of the cut-importance component ranking simpler when the system contains modules.

Example 2 (continued): Components 3 and 6 form a module.

$$A = \{3,6\}, \quad \chi(\underline{x}^A) = x_3 x_6, \quad \psi(z,\underline{x}^{A^c}) = 1 - (1 - x_1 x_2 x_4)(1 - x_1 z)(1 - x_2 x_5).$$

$$\underline{b}^{\chi 3} = (1,-1), \qquad \underline{b}^{\chi 6} = (1,-1), \qquad \underline{b}^{\psi 1} = (0,1,0,-2,1).$$

By using the concept of the dual of a coherent system (2), it is possible to develop a component ranking analogous to $>_c$, but based upon minimal paths (2) instead of minimal cuts. This ordering can be shown to be identical to the ordering induced by $I_h(\cdot;p)$ when p is sufficiently small.

5. Component Importance in Multi-State Systems

This section deals with extensions of the results of the preceding sections to systems in which both the system and its components may be in any of a finite number of states. Of course, any such increase in the sophistication of the model used to represent a real system entails both advantages and disadvantages. An obvious advantage is the increased precision with which the real system can be modeled. The disadvantages include the necessities for collecting more sophisticated data and making more complex decisions about how the model should be fitted to the system.

This extension to multi-state models is not intended to suggest that binary models are inadequate. To the contrary, in most cases they suffice quite well. However, in some instances a small increase in the number of states (say, to three or perhaps four) can result in a much improved model.

One of the main difficulties with multi-state models is the increased notational complexity. For this reason and for the reason that the number of states in a practical model must be kept small if the model is to be manageable, the following definitions will be given for ternary (three-state) systems. Whenever the extension of a definition or result to general n-state systems is unclear, an explanation will be given.

The study of multi-state systems is a relatively new area in reliability theory. Most articles in this area have dealt with generalizing particular classes of results (1), (8), (10), (11). The most general paper in the area is Barlow's (1). Let X_j denote the state of component j, $(X_j = 0,1,2,\ 1 \leq j \leq n)$. Given a collection of minimal cuts $C_1, C_2, \ldots, C_t$ which define the system structure, Barlow defines the system state $\phi(\underline{X})$ as the state of the "best" component in the "worst" min cut, i.e.,

$$\phi(\underline{X}) = \min_{1 \leq i \leq t} \{ \max_{j \in C_i} [X_j] \} .$$

Let $Z_j = I_{\{X_j \geq k\}}$ and let $\psi = I_{\{\phi(\underline{X}) \geq k\}}$. Both $\underline{Z}$ and ψ are binary, and ψ is a function only of $\underline{Z}$. Because of this property, most results about binary coherent systems have immediate generalizations

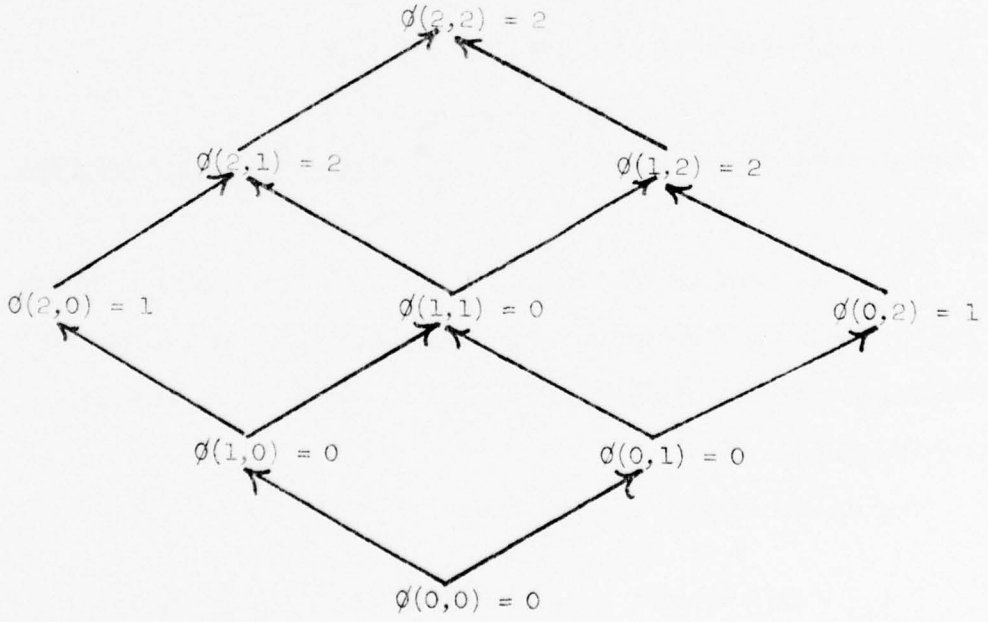
under Barlow's extended definition. However, Barlow's definition may not be sufficiently general for some systems.

Example 3. Engine Sub-System of a Light, Twin-Engine Airplane

Let $X_j = \begin{cases} 0 & \text{if engine } j \text{ siezed (prop dead in air),} \\ 1 & \text{if engine } j \text{ is "feathered",} \\ 2 & \text{if engine } j \text{ is fully functional.} \end{cases}$ $\quad j = 1,2$

$$\phi(X_1,X_2) = \begin{cases} 0 & \text{if plane cannot land under power,} \\ 1 & \text{if plane cannot maintain altitude but can land under power,} \\ 2 & \text{if plane can maintain altitude.} \end{cases}$$

In the following diagram, the component states are shown in a lattice arrangement according to the less-than-or-equal-to relation.



This ternary system cannot be fit into the framework of Barlow's definition. To accommodate such systems, a more general definition of a multi-state coherent system is proposed below.

Let $S = \{\underline{x} \in \mathbb{R}^n : x_i = 0,1,2\}$, and let $(2_i,\underline{x}) = (x_1, x_2, \ldots, x_{i-1}, 2, x_{i+1}, \ldots, x_n)$.

Definition 11: Component i is relevant if and only if $\phi(2_i,\underline{x}) \neq \phi(0_i,\underline{x})$ for some $\underline{x} \in S$. Otherwise component i is irrelevant. Component i is fully relevant if and only if $\phi(2_i,\underline{x}) \neq \phi(1_i,\underline{x})$ for some $\underline{x} \in S$ and $\phi(1_i,\underline{y}) \neq \phi(0_i,\underline{y})$ for some $\underline{y} \in S$.

Definition 12: A structure function ϕ is coherent if and only if

i) $\phi(\underline{0}) = 0$; $\phi(\underline{2}) = 2$,

ii) $\phi(\underline{x})$ is non-decreasing in $\underline{x}$,

iii) each component is relevant.

The ordered pair (N,ϕ) is called a (generalized or ternary) coherent system.

If a component is not fully relevant, then only two states are required to describe its status. Such components are permissible in a generalized coherent system to allow for a mixture of binary and ternary components.

Define the matrix $P = [p_{ij}]$ by

$$p_{ij} = \Pr\{\text{component i is in state j}\}, \qquad 1 \leq i \leq n,\ 0 \leq j \leq 2.$$

The reliability function, $h(P)$,* is defined by

$$h(P) = \Pr\{\phi(\underline{X}) \geq m\} ,$$

where $0 < m \leq 2$. (For simplicity, the dependence of $h(\cdot)$ upon m is suppressed in the notation.) For any matrix $A = [a_{ij}]$, let (k_ℓ, A) denote the matrix whose i-jth entry is given by

$$(k_\ell, A)_{ij} = \begin{cases} a_{ij} & i \neq \ell, \\ 1 & i = \ell,\ j = k, \\ 0 & i = \ell,\ j \neq k. \end{cases}$$

Definition 13: The r,s reliability importance of component i, denoted by $I_h^{r,s}(i;P)$, is given by

$$I_h^{r,s}(i;P) = h(r_i,P) - h(s_i,P) ,$$

where $r, s = 0,1,2$ and $r > s$. The 2,0 reliability importance will sometimes be simply called the reliability importance and be denoted by $I_h(i;P)$.

The r,s reliability importance of component i is the probability that the system is in state m or better given component i is in state r minus the probability that the system is in state m or better given component i is in state s. As for its binary counterpart, the generalized

*In this section and the next, whenever the vector $p \in \mathbb{R}^3$ appears in an expression normally involving the matrix P, P will be understood to be the matrix all of whose rows are equal to p.

reliability importance can be used to calculate the change in the system reliability which results from a change in the reliability of component i.

Proposition 6: Let $\delta_0, \delta_1, \delta_2 \in \mathbb{R}$ satisfy $\delta_0 + \delta_1 + \delta_2 = 0$ and $0 \leq p_{kj} + \delta_j \leq 1$, $j = 0,1,2$. Define $\hat{P} = [\hat{p}_{ij}]$

$$\hat{p}_{ij} = \begin{cases} p_{ij} & i \neq k , \\ p_{kj} + \delta_j & i = k . \end{cases}$$

Then

$$h(\hat{P}) = h(P) + \delta_2 I_h^{2,0}(k;P) + \delta_1 I_h^{1,0}(k;P)$$

Proof: For any stochastic matrix Q, by conditioning on X_k, we have

$$h(Q) = \sum_{j=0}^{2} q_{kj} h(j_k, Q).^*$$

Thus

$$\begin{aligned} & h(P) + \delta_2 I_h^{2,0}(k;P) + \delta_1 I_h^{1,0}(k;P) \\ &= \sum_{j=0}^{2} p_{kj} h(j_k, P) + \delta_2 [h(2_k, P) - h(0_k, P)] + \delta_1 [h(1_k, P) - h(0_k, P)] \\ &= \sum_{j=0}^{2} \hat{p}_{kj} h(j_k, P) \\ &= \sum_{j=0}^{2} \hat{p}_{kj} h(j_k, \hat{P}) = h(\hat{P}) . \end{aligned}$$

□

*See comment at bottom of page 3.

Definition 13: A vector $\underline{x} \in S$ is r,s critical for component i if and only if $\phi(r_i,\underline{x}) - \phi(s_i,\underline{x}) > 0$. (r, s = 0,1,2, r > s.)

Definition 14: Let $n_\phi^{r,s} = |\{\underline{x} \in S : \underline{x} \text{ is r,s critical for component i}\}|$. The r,s structural importance of component i, $I_\phi^{r,s}(i)$, is given by

$$I_\phi^{r,s}(i) = 3^{-n}\, n_\phi^{r,s}(i) .$$

Proposition 7:

i) $I_h^{2,0}(i) = I_h^{2,1}(i) + I_h^{1,0}(i)$

ii) $I_\phi^{2,0}(i) = I_\phi^{2,1}(i) + I_\phi^{1,0}(i)$

iii) $I_\phi^{r,s}(i) = I_h^{r,s}(i;\ (1/3,\ 1/3,\ 1/3))$

Proof: The proofs of (i) and (ii) are trivial. To prove (iii), note that by summing over the 3^{n-1} possible values for X_k, $k \neq i$,

$$h(j_i,\ (1/3,1/3,1/3)) = 3^{-n+1} \sum_{\substack{\underline{x}\in S \\ x_i = j}} \phi(j_i,\underline{x}) ,$$

$$= 3^{-n} \sum_{\underline{x}\in S} \phi(j_i,\underline{x}).$$

Thus

$$I_h^{2,1}(i;\ (1/3,1/3,1/3)) = 3^{-n} \sum_{\underline{x}\in S} [\phi(2_i,\underline{x}) - \phi(1_i,\underline{x})]$$

$$= 3^{-n}\, n_\phi^{2,1}(i) = I_\phi^{2,1}(i) .$$

The proof for $I_{\phi}^{1,0}(\cdot)$ is the same and the proof for $I_{\phi}^{2,0}(\cdot)$ follows from parts (i) and (ii).

□

Parts (i) and (ii) of the above result show that both importance measures decompose into the sum of $r+1,r$ importance measures. The generalized cut-importance ranking to be defined later has a similar property. In practice, it is likely that the 2,0 measures and rankings would be the most commonly used. However, the other measures and rankings can be useful in providing more detailed information about which states are most relevant in determining a given component's ranking. (See Example 4 on pages 34-35.)

Given a generalized coherent system (N,ϕ), and a partition $\mathcal{C} = (C_0, C_1, C_2)$ of N into three sets, define $\underline{x}(\mathcal{C}) \in S$ by

$$(\underline{x}(\mathcal{C}))_i = \begin{cases} 0 & i \in C_0 \\ 1 & i \in C_1 \\ 2 & i \in C_2 \end{cases}.$$

The function $\underline{x}(\mathcal{C})$ shows how any partition $\mathcal{C}$ determines the states of all the components.

Definition 15: A partition $\mathcal{C} = (C_0, C_1, C_2)$ of N is a cut if and and only if $\phi(\underline{x}(\mathcal{C})) < m$. A cut $\mathcal{C}$ is a minimal cut if and only if $\phi(\underline{y}) \geq m$ for all $\underline{y} \in S$ such that $\underline{y} \geq \underline{x}(\mathcal{C})$, $\underline{y} \neq \underline{x}(\mathcal{C})$.

Example 3 (continued):

m = 1		m = 2	
Min cuts:	$\mathcal{C}^1 = (E, \{1,2\}, E)$	Min cuts:	$\mathcal{C}^1 = (\{1\}, E, \{2\})$
			$\mathcal{C}^2 = (\{2\}, E, \{1\})$
			$\mathcal{C}^3 = (E, \{1,2\}, E)$

(E denotes the empty set.)

While it is in principle possible to develop a complete cut-importance ranking for generalized coherent systems, in practice the calculation of the entire generalized $\underline{b}^{(\cdot)}$ vector for each component is too complex to be feasible. However, a partial ordering of the components which involves very few calculations can be developed by generalizing Proposition 4 and Corollary 1 appropriately. First, the notions of the size of a partition and the union of partitions must be defined.

Definition 16: The size of a partition $\mathcal{C} = (C_0, C_1, C_2)$, denoted by $z(\mathcal{C})$, is $\alpha_0|C_0| + \alpha_1|C_1|$. ($\alpha_0$, α_1 are arbitrary constants satisfying $\alpha_0 > \alpha_1 > 0$.)

The roles of the constants α_0, α_1 are discussed on page 31.

Definition 17: Let $\mathcal{T}^1, \mathcal{T}^2, \ldots, \mathcal{T}^j$ be partitions of N where

$$\mathcal{T}^i = (T_0^i, T_1^i, T_2^i) .$$

The union of $\mathcal{T}^1, \ldots, \mathcal{T}^j$ is the partition

$$(V_0, V_1 - V_0, V_2 - V_1 - V_0)$$

where $V_k = \bigcup_{i=1}^{j} T_k^i$.

Definition 18: Consider a ternary coherent system with minimal cuts $\mathcal{C}^i = (C_0^i, C_1^i, C_2^i)$, $i = 1, 2, \ldots, t$. For each component k, let

$$e_k^{r+1,r} = \min_{1 \le i \le t} \{z(\mathcal{C}^i) : k \in C_r^i\} - \alpha_r ,$$

and let

$$f_k^{r+1,r} = |\{\mathcal{C}^i : k \in C_r^i,\ z(\mathcal{C}^i) - \alpha_r = e_k^{r+1,r},\ 1 \le i \le t\}| .$$

(By convention, $e_k^{r+1,r} = +\infty$ if $k \notin C_r^i$, $1 \le i \le t$.) For all $r,s = 0,1,2$ such that $r > s$, define

$$e_k^{r,s} = \min_{s \le u < r} \{e_k^{u+1,u}\}$$

and

$$f_k^{r,s} = \sum_{\substack{s \le u < r \\ e_k^{u+1,u} = e_k^{r,s}}} f_k^{u+1,u}$$

Component ℓ is <u>more r,s cut-important</u> than component k, denoted $\ell >_c^{r,s} k$, if and only if either

$$\text{i)} \quad e_\ell^{r,s} < e_k^{r,s} ,$$

or

$$\text{ii)} \quad e_\ell^{r,s} = e_k^{r,s} \quad \text{and} \quad f_\ell^{r,s} > f_k^{r,s} .$$

The 2,0 cut-importance ranking will sometimes be simply called the <u>cut-importance ranking</u> and be denoted by $>_c$.

As in the binary case, each of the r,s importance rankings is consistent with the ranking induced by the corresponding r,s importance measure. To be more specific, let $\underline{p}(\epsilon) = (\epsilon^{\alpha_0}, \epsilon^{\alpha_1}, 1 - \epsilon^{\alpha_0} - \epsilon^{\alpha_1})$. As ϵ approaches zero, $\underline{p}(\epsilon)$ puts almost all its mass on the best state, state 2. Of the mass left over, the ratio of the mass put on state 0 to that put on state 1 approaches zero. Thus the parameters α_0 and α_1 give the relative weights put on components in state zero versus components in state 1 in the cut importance ranking and also determine the relative likelihoods of a component partially failing (state 1) and fully failing (state zero).

<u>Theorem 3</u>: For ϵ sufficiently close to zero, the component ranking induced by $I_h^{r,s}(\cdot;\underline{p}(\epsilon))$ is consistent with the r,s cut-importance ranking ($r > s$).

Proof: $I_h^{r+1,r}(k;\underline{p}(\epsilon)) = h((r+1)_k,\underline{p}(\epsilon)) - h(r_k,\underline{p}(\epsilon))$

$$= 1 - \Pr\{\bigcup_{i=1}^{t} E_i | X_k = r+1\} - [1 - \Pr\{\bigcup_{i=1}^{t} E_i | X_k = r\}],$$

where $E_i = \{\underline{X} \le \underline{x}(\mathcal{C}^i)\}$. Thus by the inclusion-exclusion principle,

$$I_h^{r+1,r}(k;\underline{p}(\epsilon)) = \sum_{i=1}^{t} (-1)^{i-1} [S_i^r - S_i^{r+1}] , \qquad (3)$$

where

$$S^r - S_i^{r+1} = \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} [\Pr\{\bigcap_{\ell=1}^{i} E_{j_\ell} | X_k = r\} - \Pr\{\bigcap_{\ell=1}^{i} E_{j_\ell} | X_k = r+1\}]$$

$$= \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} \Pr\{\bigcap_{\ell=1}^{i} E_{j_\ell} | X_k = r\}$$

$$r = \min_{1 \le \ell \le i} \{x_k(\mathcal{C}^{j_\ell})\}$$

$$= \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} \Pr\{X_w \le \min_{1 \le \ell \le i} \{x_w(\mathcal{C}^{j_\ell})\}, 1 \le w \le n | X_k = r\}$$

$$r = \min_{1 \le \ell \le i} \{x_k(\mathcal{C}^{j_\ell})\}$$

$$= \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} \Pr\{\underline{X} \le \underline{x}(\mathcal{C}) | X_k = r\}$$

$$k \in D_r$$

$$= \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} \epsilon^{-\alpha_r} \cdot \epsilon^{\alpha_0 |D_0|} \cdot (\epsilon^{\alpha_0} + \epsilon^{\alpha_1})^{|D_1|}$$

$$k \in D_r$$

where $\mathcal{D} = (D_0, D_1, D_2)$ is the union of $\mathcal{C}^{j_1}, \mathcal{C}^{j_2}, \ldots, \mathcal{C}^{j_i}$. By the definitions of e_k and f_k^*, the lowest order term in this polynomial expression for $S_1^r - S_1^{r+1}$ is $f_k \epsilon^{e_k}$. Thus

$$S_1^r - S_1^{r+1} = f_k \epsilon^{e_k} + o(\epsilon^{e_k}) .$$

Next we show that $S_i^r - S_i^{r+1} = o(\epsilon^{e_k})$ for all $i \geq 2$. Let $\mathcal{D}$ be the union of any i minimal cuts $\mathcal{C}^{j_1}, \ldots, \mathcal{C}^{j_i}$ satisfying $k \in D_r$. Then

$$|D_0| \geq |C_0^{j_1}|$$

and (4)

$$|D_0| + |D_1| \geq |C_0^{j_1}| + |C_1^{j_1}| .$$

The above two inequalities cannot simultaneously hold as equalities because then $|D_0| = |C_0^{j_1}|$, which implies that $D_0 = C_0^{j_1}$ and $|D_1| = |C_1^{j_1}|$, which implies that $D_1 = C_1^{j_1}$. Thus $\mathcal{D} = \mathcal{C}^{j_1}$ and so $\underline{x}(\mathcal{D}) = \underline{x}(\mathcal{C}^{j_1})$. Now as an immediate consequence of the definition of $\mathcal{D}$, $\underline{x}(\mathcal{D}) \leq \underline{x}(\mathcal{C}^{j_2})$, and so $\underline{x}(\mathcal{C}^{j_1}) \leq \underline{x}(\mathcal{C}^{j_2})$. Furthermore, $\mathcal{C}^{j_1} \neq \mathcal{C}^{j_2}$, so the inequality must be strict in at least one coordinate. But this contradicts the assumption that the cut $\mathcal{C}^{j_1}$ is minimal, and so at least one of the inequalities in (4) must be strict.

Now the lowest power in the polynomial expression for $S_i^r - S_i^{r+1}$ is $\alpha_0|D_0| + \alpha_1|D_1| - \alpha_r$. But

---

*For notational simplicity the superscript r+1,r which should appear on e_k, f_k, $>_c$, and I_h will be dropped in the remainder of the proof.

$$\alpha_0|D_0| + \alpha_1|D_1| - \alpha_r = (\alpha_0 - \alpha_1)|D_0| + \alpha_1(|D_0| + |D_1|) - \alpha_r$$
$$> (\alpha_0 - \alpha_1)|C_0^{j_1}| + \alpha_1(|C_0^{j_1}| + |C_1^{j_1}|) - \alpha_r$$
$$> \alpha_0|C_0^{j_1}| + \alpha_1|C_1^{j_1}| - \alpha_r \geq e_k .$$

Thus

$$S_i^r - S_i^{r+1} = o(\epsilon^{e_k}) .$$

Thus, by equation (2)

$$I_h(k;\underline{p}(\epsilon)) = f_k \cdot \epsilon^{e_k} + o(\epsilon^{e_k}) . \tag{5}$$

Now assume that $\ell >_c k$. If $e_\ell < e_k$, then by equation (5)

$$I_h(\ell;\underline{p}(\epsilon)) - I_h(k;\underline{p}(\epsilon)) = f_\ell \cdot \epsilon^{e_\ell} + o(\epsilon^{e_\ell}) .$$

Thus for ϵ sufficiently close to zero this expression is positive and so the two orderings of ℓ and k are identical in this case.

If $e_\ell = e_k$ and $f_\ell > f_k$, then again by equation (5)

$$I_h(\ell;\underline{p}(\epsilon)) - I_h(k;\underline{p}(\epsilon)) = (f_\ell - f_k)\epsilon^{e_\ell} + o(\epsilon^{e_\ell}) .$$

Thus in this case, also, the two orderings are consistent for ϵ sufficiently small. This establishes the theorem for all the $r+1,r$ orderings. To establish the result for any r,s ordering* note that

$$I_n^{r,s}(k;\underline{p}(\epsilon)) = \sum_{u=s}^{r-1} I_h^{u+1,u}(k;\underline{p}(\epsilon)) \ .$$

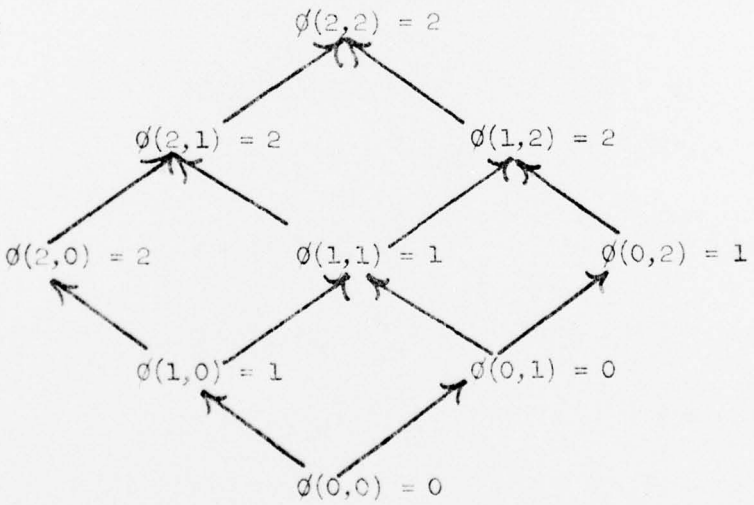
(See Proposition 7, part (i).) Combining this result with equation (5),

$$I_h^{r,s}(k;\underline{p}(\epsilon)) = f_k^{r,s}\ \epsilon^{e_k^{r,s}} + o(\epsilon^{e_k^{r,s}}) \ .$$

The remainder of the proof is identical to the $r+1,r$ case.

□

Example 4:



*For ternary systems, the only $r+1,r$ orderings are the 2-1 ordering and the 1-0 ordering. The only other r,s ordering is the 2-0 ordering. The notation in the proof and the definition of r,s cut-importance has been kept more general so that the extensions to general n-state systems can be more readily understood.

Consider the case where $m = 2$, $\alpha_0 = 2$, $\alpha_1 = 1$.

Min cuts: $\mathcal{C}^1 = (E, \{1,2\}, E)$

$\mathcal{C}^2 = (\{1\}, E, \{2\})$

$e_1^{2,1} = 1$; $e_2^{2,1} = 1$ Components 1, 2 are not comparable under the 2, 1 ranking.

$f_1^{2,1} = 1$; $f_2^{2,1} = 1$

$e_1^{1,0} = 0$; $e_2^{1,0} = +\infty$ $1 >_c^{1,0} 2$

$f_1^{1,0} = 1$; $f_2^{1,0} = 0$

$e_1^{2,0} = 0$; $e_2^{2,0} = 1$

$1 >_c^{2,0} 2$

$f_1^{2,0} = 1$; $f_2^{2,0} = 1$

$$h(P) = 1 - (p_{10} + p_{11})(p_{20} + p_{21}) - p_{10} + p_{10}(p_{20} + p_{21})$$

$$= 1 - p_{11}(p_{20} + p_{21}) - p_{10}.$$

$I_h^{2,1}(1;\underline{p}(\epsilon)) = \epsilon^2 + \epsilon.$ $\quad I_h^{2,1}(2;\underline{p}(\epsilon)) = \epsilon .$

$I_h^{1,0}(1;\underline{p}(\epsilon)) = 1 - \epsilon - \epsilon^2.$ $\quad I_h^{1,0}(2;\underline{p}(\epsilon)) = 0 .$

$I_h^{2,0}(1;\underline{p}(\epsilon)) = 1 .$ $\quad I_h^{2,0}(2;\underline{p}(\epsilon)) = \epsilon .$

Thus component 1 is more important than component 2 in an overall sense (i.e. according to the 2,0 cut importance ranking). Moreover, the 2,1 and 1,0 rankings of the components show that it is the state 1 to state 0 transition of the components which determines the 2,0 ranking here.

As was the case for binary systems, analogous results based upon minimal paths can be developed for ternary systems composed of very unreliable components.

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TR 183 A COMPLETE IMPORTANCE RANKING FOR COMPONENTS OF BINARY COHERENT SYSTEMS, WITH EXTENSIONS TO MULTI-STATE SYSTEMS

Given a system composed of many components, a question of considerable interest is which components are most relevant or crucial to the proper functioning of the system. In response to this question, a number of importance measures and rankings have been proposed. This paper investigates a new ranking and compares it to existing rankings, principally the ranking induced by the Birnbaum reliability importance measure. The new ranking is based upon minimal cuts and provides a complete ordering of all the system's components relative to their importance to the system reliability. This ranking has three main points in its favor, (i) the calculations involved require only readily obtainable information; (ii) the calculations are usually quite simple; and (iii) the ranking is designed for use with systems consisting of highly reliable components, the most common case. The final section of the paper deals with extensions of importance measures and rankings to systems in which both the system and its components may be in any of a finite number of states. Many of the results about importance measures and rankings for binary systems established in preceding sections are shown to extend to the more sophisticated multi-state systems. Also, the multi-state importance measures and rankings are shown to be decomposable into a number of sub-importance measures and rankings.